Full counting statistics and conditional evolution in a nanoelectromechanical system

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We study theoretically the full distribution of transferred charge in a tunnel junction (or quantum point contact) coupled to a nanomechanical oscillator, as well as the conditional evolution of the oscillator. Even if the oscillator is very weakly coupled to the tunnel junction, it can strongly affect the tunneling statistics and lead to a highly non-Gaussian distribution. Conversely, given a particular measurement history of the current, the oscillator energy distribution may be localized and highly nonthermal. We also discuss non-Gaussian correlations between the oscillator motion and tunneling electrons; these show that the tunneling back-action cannot be fully described as an effective thermal bath coupled to the oscillator.

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I. INTRODUCTION

The possibility to observe the quantum mechanics of a macroscopic object has sparked significant interest in nanoelectromechanical systems (NEMS), which consist of a mechanical oscillator coupled to a mesoscopic conductor. In recent experiments, the oscillator motion has been measured with near quantum-limited precision using the conductor as a detector[1–3] and cooling of the oscillator by quantum back-action has been observed.[4] In these experiments it is the current noise of the conductor (i.e., the second moment of current fluctuations) that is used to measure position fluctuations of the oscillator. The effect of the oscillator on the current noise has been well studied theoretically, leading to new understanding of back-action and quantum dissipation in NEMS.[5–11] However, much more information lies in the *full* probability distribution of transmitted charge through the conductor or the full counting statistics (FCS).[12] In addition to being of theoretical interest, FCS is an experimentally accessible quantity and the third moment was recently measured in a tunnel junction.[13] Still more information may be gained by considering the conditional evolution: given a particular measurement history for the current, what can we say about the state of the oscillator?

In this paper, we study the full statistics of a tunnel junction (or quantum point contact) coupled to a nanomechanical oscillator, a system recently realized in experiment.[1,2] This system is a prime candidate for measuring FCS in NEMS since the intrinsic shot noise can dominate over other noise sources, making it feasible to measure the higher moments. Starting from a microscopic fully quantum model, we calculate the FCS of tunneled charge as well as the conditional evolution of the oscillator and find several surprises that would not be apparent in a study of the noise alone. Despite weak oscillator-conductor coupling, we find that the oscillator can strongly enhance the third and higher moments of the FCS, leading to a markedly non-Gaussian distribution. This results from long-lived energy fluctuations in the high-Q oscillator, which allow correlations between the oscillator motion and tunneling electrons to accumulate up to the ring-down time of the oscillator, overwhelming the weak coupling strength and dominating the FCS. Further, even though the conductor couples linearly to the oscillator position, the oscillator state conditioned on a particular measurement of current can be highly localized in energy. Finally, we discuss non-Gaussian correlations between the current and back-action force on the oscillator that are not captured by treating the tunnel junction as an effective thermal bath. These correlations arise from the random momentum kicks imparted to the oscillator by individual tunneling electrons, which cause one-half of the back-action to be correlated with tunneling. The non-Gaussian correlations lead to signatures in measurable quantities such as the current noise; thus, measuring the current noise could provide a way to distinguish quantum back-action from the effects of an equilibrium bath.

Note that FCS were studied previously in a very different type of NEMS, a charge shuttle.[14,15] Conditional evolution in NEMS was studied using a quantum optics approach,[16,17] but these studies miss key features arising in our microscopically derived model. The average current and current noise of the NEMS studied here were addressed in Refs. 8–11; unlike these works, we present an exact solution of the master equation and study the FCS.

II. MASTER EQUATION AND ITS SOLUTION

The Hamiltonian of the coupled system is $H=H_{\rm osc}+H_{\rm leads}+H_T$, where $H_{\rm osc}$ describes a harmonic oscillator of mass M and frequency Ω including dissipation due to an equilibrium thermal bath at temperature T_0.[18] Electrons in the leads are described by $H_{\rm leads}=\Sigma_{\alpha,k}\varepsilon_k c^\dagger_{\alpha k}c_{\alpha k}-eV\hat{m}$, where $c_{\alpha k}$ annihilates an electron in lead $\alpha=L,R$, V is the junction bias voltage, and the operator $\hat{m}$ counts the number of tunneled electrons. H_T describes electron tunneling for the experimentally relevant case of weak oscillator-junction coupling,[8–11]

$$H_T=\frac{\tau_0+e^{i\eta}\tau_1\hat{x}}{2\pi\Lambda}\sum_{k,k'}(Y^\dagger c^\dagger_{Rk}c_{Lk'}+\text{H.c.}), \tag{1}$$

where Λ is the lead density of states, η describes the dependence of the transmission phase on the oscillator position $\hat{x}$, and $Y^\dagger$ is the raising operator associated with $\hat{m}$, e.g., $[\hat{m},Y^\dagger]=Y^\dagger$. We focus on an inversion symmetric system in which η vanishes.[8]

We describe the system using a reduced density matrix $\hat{\rho}(t)$ tracking the state of the oscillator and m, the number of

tunneled electrons. In the weak tunneling regime of interest, the off-diagonal (in m) elements of $\hat{\rho}$ decouple from the diagonal elements. Since our aim is to calculate the statistics of m, we need only to consider the diagonal elements, $\hat{\rho}(m;t) = \langle m|\hat{\rho}(t)|m\rangle$. Treating H_T perturbatively and making a standard Markov approximation which requires $eV \gg \hbar\Omega$, we derive a master equation for $\hat{\rho}(m;t)$.[8] Fourier transforming in m, $\hat{\rho}(\chi;t) = \Sigma_{m=-\infty}^{\infty} e^{i\chi m}\hat{\rho}(m;t)$, the equation reads

$$\partial_t\hat{\rho}(\chi;t) = -\frac{i}{\hbar}[H_0,\hat{\rho}] - \frac{i\gamma}{2\hbar}[\hat{x},\{\hat{p},\hat{\rho}\}] - \frac{D}{\hbar^2}[\hat{x},[\hat{x},\hat{\rho}]] + \frac{(e^{i\chi}-1)}{\tau_1^2}\left\{\frac{2D_1}{\hbar^2}(\tau_0+\tau_1\hat{x})\hat{\rho}(\tau_0+\tau_1\hat{x}) + \frac{i\gamma_1}{2\hbar}[\tau_0\tau_1(\hat{p}\hat{\rho}-\hat{\rho}\hat{p}) + \tau_1^2(\hat{p}\hat{\rho}\hat{x}-\hat{x}\hat{\rho}\hat{p})]\right\}. \quad (2)$$

Here, H_0 describes the coherent dynamics of the oscillator and the total damping and diffusion coefficients are $\gamma = \gamma_0 + \gamma_1$ and $D = D_0 + D_1$. The coefficients γ_0 and $D_0 = (M\gamma_0\hbar\Omega/2)\coth(\hbar\Omega/2T_0)$ are associated with the equilibrium bath ($k_B = 1$), while γ_1 and D_1 describe back-action damping and diffusion due to the tunnel junction. Taking the electronic temperature in the leads to be much less than eV,[19] these are given by $\gamma_1 = \hbar\tau_1^2/2\pi M$ and $D_1 = M\gamma_1 T_1$, where the effective temperature due to the tunnel junction is $T_1 = eV/2$.[8,9] Note that if we average over m (i.e., set $\chi = 0$), Eq. (2) reduces to the quantum Brownian motion master equation for an oscillator coupled to two effective thermal baths consisting of the environment and the tunnel junction.[18] Conversely, tracing over the oscillator degrees of freedom yields the generating function for the FCS,

$$\Phi(\chi;t) = \mathrm{tr}[\hat{\rho}(\chi;t)] = \sum_m e^{i\chi m}P(m;t), \quad (3)$$

where $P(m;t)$ is the probability that m electrons have tunneled in time interval t. Note that the trace of $\hat{\rho}(t)$ over *all* degrees of freedom is $\Sigma_m \mathrm{tr}[\hat{\rho}(m;t)] = 1$.

The above model has been used to study the average current and noise;[8–11] here we present its exact solution and use it to study FCS and conditional evolution. To work with Eq. (2), we first express the reduced density matrix in its Wigner representation,

$$W(x,p) = \frac{1}{\pi\hbar}\int dy\langle x+y|\hat{\rho}|x-y\rangle e^{-2iyp/\hbar}. \quad (4)$$

In terms of the Wigner function, Eq. (2) may be written as

$$\partial_t W(x,p;\chi;t) = (\mathcal{L}_{\mathrm{cl}} + \mathcal{L}_{\mathrm{q}})W, \quad (5)$$

where the evolution is described by two Liouvillian operators,

$$\mathcal{L}_{\mathrm{cl}} = -\frac{p}{M}\partial_x + M\Omega^2 x\partial_p + \gamma\partial_p\cdot p + D\partial_p^2 + (e^{i\chi}-1)\Gamma(1+\lambda x)^2, \quad (6a)$$

$$\mathcal{L}_{\mathrm{q}} = (e^{i\chi}-1)\left[\frac{\gamma_1\partial_p\cdot p + D_1\partial_p^2}{2} + \frac{\hbar^2}{4\pi M}\tau_1(\tau_0+\tau_1 x)\partial_x\right], \quad (6b)$$

with the tunneling rate for the oscillator at $x = 0$ given by $\Gamma = \tau_0^2 eV/2\pi\hbar = 2D_1\tau_0^2/\hbar^2\tau_1^2$. The Liouvillian operator $\mathcal{L}_{\mathrm{cl}}$ describes the effectively classical evolution of the system: the first line of Eq. (6a) corresponds to a classical Fokker-Planck equation for the oscillator coupled to two effective equilibrium baths, the environment and the junction; the second line describes tunneling as a classical Poisson process characterized by a rate $\Gamma(t)$ that depends on the instantaneous oscillator position $x(t)$. In contrast, $\mathcal{L}_{\mathrm{q}}$ accounts for quantum corrections to the effectively classical evolution. The χ-dependent terms involving γ_1 and D_1 in Eq. (6b) describe *conditional* damping and diffusion; these terms represent back-action that is correlated with tunneling. Conditional back-action arises because each tunneling electron imparts a random momentum kick to the oscillator, implying that the momentum kicks are correlated in time with tunneling events, and shows that the back-action of the tunnel junction is *not* fully described as an effective equilibrium bath. This is discussed in detail in Sec. V. Note that when we add the back-action terms in $\mathcal{L}_{\mathrm{q}}$ to those in $\mathcal{L}_{\mathrm{cl}}$, we find that exactly *half* of the total back-action is conditional (i.e., includes the factor $e^{i\chi}$). The other half of the back-action is uncorrelated with tunneling and cannot be understood in terms of momentum kicks imparted by tunneling electrons. We thus have the surprising conclusion that even during periods where no electrons tunnel, there is still back-action diffusion and damping. Heuristically, even if no electrons tunnel, we nonetheless gain information about the oscillator and therefore there must be back-action. Finally, the remaining terms in Eq. (6b) are also quantum in nature and arise from the difference between two tunneling processes involving absorption or emission of a phonon of energy $\hbar\Omega$. In particular, the last term $\propto\tau_1^2 x\partial_x W$ does not vanish when we trace over the oscillator degrees of freedom and thus represents a quantum correction to the average tunneling rate[8,9] [cf. Eq. (11) below]. The same correction is obtained from a direct calculation of the tunneling rate using Fermi's golden rule.

Equation (5) may be solved exactly for the physical initial conditions of a thermal oscillator state. Such a state is Gaussian and remains Gaussian under Eq. (5) for all times. We also take $m = 0$ at time $t = 0$ since this is when we start counting tunneled electrons. Thus, the Wigner function may be written in the form

$$W(x,p;\chi;t) = \frac{e^{\phi}e^{-[V_p(x-\bar{x})^2 + V_x(p-\bar{p})^2 - 2V_{xp}(x-\bar{x})(p-\bar{p})]/2(V_xV_p - V_{xp}^2)}}{2\pi\sqrt{V_xV_p - V_{xp}^2}}, \quad (7)$$

where we have scaled all quantities by the natural units of the zero-point motion of the oscillator, $\Delta x_0 = \sqrt{\hbar/2M\Omega}$ and $\Delta p_0 = \sqrt{M\hbar\Omega/2}$. The state is fully characterized at all times by its means, $\bar{x}$ and $\bar{p}$, its variances, V_x and V_p, its covariance, V_{xp}, and its normalization, e^{ϕ}. These six Gaussian parameters depend on both χ and t and satisfy simple ordinary differen-

tial equations which follow directly from Eq. (5). First, the means satisfy

$$\partial_t \bar{x}(\chi;t) = \Omega \bar{p} + \gamma_1 (e^{i\chi} - 1)\left(\bar{x} + \frac{1}{\lambda}\right)\left(\frac{2T_1}{\hbar\Omega} V_x - \frac{1}{2}\right), \quad (8a)$$

$$\partial_t \bar{p}(\chi;t) = -\Omega \bar{x} - \gamma \bar{p} + \gamma_1 (e^{i\chi} - 1)\left[\frac{2T_1}{\hbar\Omega} V_{xp}\left(\bar{x} + \frac{1}{\lambda}\right) - \frac{\bar{p}}{2}\right], \quad (8b)$$

where we have again scaled the position and momentum by Δx_0 and Δp_0 and defined the dimensionless coupling strength $\lambda = \Delta x_0 \tau_1 / \tau_0$. The χ dependence of $\bar{x}$ and $\bar{p}$ encodes correlations between the oscillator motion and m. For example, one can easily show that the irreducible correlation between x and the nth moment of m is $\langle\langle x m^n \rangle\rangle = (-i)^n \frac{\partial^n \bar{x}}{\partial \chi^n}$. Next, the variances and covariance (also scaled by Δx_0 and Δp_0) satisfy

$$\partial_t V_x(\chi;t) = 2\Omega V_{xp} + \gamma_1 (e^{i\chi} - 1) V_x \left(\frac{2T_1}{\hbar\Omega} V_x - 1\right), \quad (9a)$$

$$\partial_t V_p(\chi;t) = -2\Omega V_{xp} - 2\gamma_0 \left(V_p - \frac{2\tilde{T}_0}{\hbar\Omega}\right) - 2\gamma_1 \left(V_p - \frac{2T_1}{\hbar\Omega}\right) - \gamma_1 (e^{i\chi} - 1)\left(V_p - \frac{2T_1}{\hbar\Omega}(1 + V_{xp}^2)\right), \quad (9b)$$

$$\partial_t V_{xp}(\chi;t) = \Omega (V_p - V_x) - \gamma V_{xp} + \gamma_1 (e^{i\chi} - 1) V_{xp}\left(\frac{2T_1}{\hbar\Omega} V_x - 1\right), \quad (9c)$$

where $\tilde{T}_0 = (\hbar\Omega/2)\coth(\hbar\Omega/2T_0)$. Again, the χ dependence of these parameters describes correlations between x^2, p^2, or xp and moments of m. Finally, the parameter ϕ satisfies

$$\partial_t \phi = (e^{i\chi} - 1)\left\{\Gamma[1 + 2\lambda\bar{x} + \lambda^2(\bar{x}^2 + V_x)] - \frac{\gamma_1}{2}\right\} \quad (10)$$

and is directly connected to the FCS as will be discussed in Sec. III.

Equations (8a), (8b), (9a)–(9c), and (10) have simple analytic solutions in the limit of long times and may be solved numerically for all times to arbitrary precision. Before using the equations to study the FCS and conditional evolution, we emphasize an important difference from previous treatments of conditional evolution in NEMS: the evolution of the variances is conditional, as seen directly from the χ-dependent terms in Eq. (9). This is in stark contrast to the standard treatment where the variances evolve independently of tunneling.[16,17] This is partly due to the conditional back-action diffusion in $\mathcal{L}_q$ discussed above [cf. Eq. (6b)], which implies that momentum fluctuations of the oscillator are correlated with fluctuations in m and leads to the conditional terms in Eq. (9b). However, we also find conditional terms in Eq. (9a) that arise from the *classical* part of Eq. (5) described by $\mathcal{L}_{cl}$. This is because we start with the linear x dependence of the tunneling *amplitude* in Eq. (1), and it follows that the tunneling *rate* has both linear and quadratic x dependences,

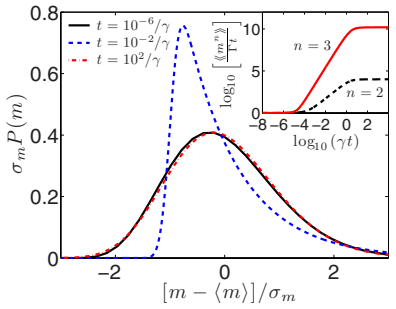

FIG. 1. (Color online) Main: $P(m;t)$ at three times. We have shifted m by its average $\langle m(t)\rangle$ and scaled by its standard deviation $\sigma_m(t)$. Inset: oscillator enhancement of the variance (dashed) and skewness (solid) of $P(m;t)$ versus time. We have taken $T_0 = T_1 = 10^4 \hbar\Omega$, $\lambda = 10^{-3}$, $\tau_0 = 0.2$, and $\gamma_0 = 10^{-5}\Omega$. These values are based on the experiment in Ref. 1, except that we decreased the intrinsic tunneling strength τ_0 to operate in the tunneling regime and increased the coupling strength λ to clearly illustrate our results. For these values, the fractional change in the average current due to the oscillator is $\lambda^2\langle\langle x^2\rangle\rangle = 0.02$.

$$\tilde{\Gamma}(x) = \Gamma[1 + 2\lambda x + \lambda^2 x^2] - \frac{\gamma_1}{2}. \quad (11)$$

Standard treatments of conditional evolution neglect the quadratic dependence, which in our case is inconsistent with the starting Hamiltonian.[20] We stress that the conditional (i.e., χ dependent) and unconditional (i.e., χ independent) terms in Eq. (9) appear at the same order in the coupling strength λ; there is no *a priori* reason to keep one effect and not the other. The results presented below are contingent on the conditional evolution of the variances.

III. FULL COUNTING STATISTICS

It follows directly from Eq. (3) that the generating function for the FCS is given by the Gaussian parameter ϕ via $\Phi(\chi;t) = e^{\phi(\chi;t)}$. From Eq. (10) we see that if the average and variance of the oscillator were simply constants, then tunneling electrons would obey Poisson statistics with an effective tunneling rate given by $\langle\tilde{\Gamma}(x)\rangle$, obtained from Eq. (11). However, the oscillator position is correlated with tunneling electrons; this correlation enters Eq. (10) through the χ dependence of $\bar{x}$ and V_x and leads to deviations from Poisson statistics. From Eqs. (3) and (10), we obtain $P(m;t)$, shown at several times in Fig. 1. Even for weak coupling, i.e., $\lambda^2\langle\langle x^2\rangle\rangle \ll 1$, the oscillator can have a dramatic effect on $P(m;t)$ at what we call intermediate times, $t^* \lesssim t \lesssim 1/\gamma$, causing it to become highly non-Gaussian. Here,

$$t^* \sim \frac{1}{\Gamma}\left(\frac{\hbar\Omega}{\lambda^2 T}\right)^2, \quad (12)$$

and $T = D/M\gamma$ is the net effective temperature of the oscillator due to both the tunnel junction and the thermal environment. We emphasize that in the relevant limit of a high-Q oscillator, the time scale $1/\gamma$ is much larger than $1/\Gamma$ and thus many electrons have tunneled even for intermediate times.

The significant modification of the FCS is due to the seemingly weak dependence of the current on x^2 [cf. Eq. (11)]. To see this, it is useful to consider the first few cumulants of m. From Eqs. (3) and (10), these satisfy (x is again scaled by Δx_0)

$$\partial_t \langle\langle m^2 \rangle\rangle = \partial_t \langle m \rangle + 2\Gamma[2\lambda \langle\langle xm \rangle\rangle + \lambda^2 \langle\langle x^2 m \rangle\rangle], \quad (13a)$$

$$\partial_t \langle\langle m^3 \rangle\rangle = \partial_t \langle m \rangle + 3\Gamma\{2\lambda[\langle\langle xm \rangle\rangle + \langle\langle xm^2 \rangle\rangle] + \lambda^2[\langle\langle x^2 m \rangle\rangle + \langle\langle x^2 m^2 \rangle\rangle + \langle\langle xm \rangle\rangle^2]\}, \quad (13b)$$

where all of correlations depend on t. The first term in each equation corresponds to Poisson statistics, in which all cumulants would be equal to $\langle m \rangle$. Correlations of x and x^2 with m emerge naturally in the cumulants due to the x dependence of the tunneling rate in Eq. (11) because $m(t) = \int_0^t dt' \tilde{\Gamma}[x(t')]$. As indicated in Eq. (13), this allows oscillator fluctuations to affect the variance and skewness of m; since x and x^2 are positively correlated with m, the cumulants will be increased by these correlations. Similar correlations appear in the higher moments. These correlations can strongly affect the cumulants due to the slow decay of energy fluctuations in the oscillator, as we now discuss.

Consider Eq. (13a). From the x^2 term in $\tilde{\Gamma}(x)$, fluctuations in x^2 will lead to fluctuations in m. Thus, the last term in Eq. (13a) leads to a term $2\Gamma^2\lambda^4 \int_0^t \int_0^t dt_1 dt_2 \langle\langle x^2(t_1) x^2(t_2) \rangle\rangle$ in the variance. The factor λ^4 is small due to weak coupling; however, the x^2 autocorrelation in the integrand is proportional to an energy autocorrelation (up to insignificant rapidly oscillating terms). This contribution initially scales as $(T/\hbar\Omega)^2$ and decays on the very slow time scale of the oscillator ringdown time, $1/\gamma$. Thus, long-lived energy fluctuations in the high-Q oscillator allow its influence to build up, eventually overcoming the weak-coupling strength and dominating the FCS. This enhancement occurs when the last term in Eq. (13a) dominates the first, requiring $\Gamma(\lambda^2 T/\hbar\Omega)^2/\gamma \gg 1$. This condition can be satisfied even when the oscillator contribution to the average current $e\langle\tilde{\Gamma}(x)\rangle$ is small, as the ratio Γ/γ is typically large (e.g., $\Gamma/\gamma \sim 10^8$ in Ref. 1). Further, this same condition ensures $t^* \ll 1/\gamma$ from Eq. (12), resulting in non-Gaussian FCS over a wide range of times.

If the condition for enhancement is met, then the effect is even greater for higher cumulants. For example, Eq. (13b) contains a term proportional to $\langle\langle x^2(t) m^2(t) \rangle\rangle$. This leads to an oscillator-dependent term in the skewness similar to that in the variance, with an additional factor of m resulting in an extra factor of $\Gamma\lambda^2 x^2$ and an extra time integral. We obtain a three-time x^2 autocorrelation which initially scales as $(T/\hbar\Omega)^3$ and decays on the time scale $1/\gamma$, compensating for the extra factor of weak coupling. In general, we find that the maximum enhancement for the nth cumulant is roughly

$$\langle\langle m^n(t) \rangle\rangle \sim (\Gamma\lambda^2 Tt/\hbar\Omega)^n \quad (14)$$

for times $t^* \ll t \ll 1/\gamma$. This can be seen directly from Eq. (11) by assuming that fluctuations in m are dominated by x^2 fluctuations for this range of times.

Figure 1 shows that $P(m;t)$ is skewed only for intermediate times $t^* \lesssim t \lesssim 1/\gamma$; for short and long times the distribution is nearly Gaussian. The enhancement of cumulants compared to their Poisson values [i.e., $\langle\langle m^n(t) \rangle\rangle = \Gamma t$ with no oscillator] is shown in the inset of Fig. 1. For short times, the effects of the weakly coupled oscillator have not yet built up and we obtain the Poisson statistics of the uncoupled tunnel junction. For long times $t \gg 1/\gamma$, the contribution to $\langle\langle m^n \rangle\rangle$ from x^2 fluctuations simply scales as t (and not as t^n), as t is now much longer than the lifetime of a typical oscillator energy fluctuation. For long times the oscillator still enhances $\langle\langle m^n \rangle\rangle$ by a factor $(\Gamma/\gamma)^{n-1}(\lambda^2 T/\hbar\Omega)^n$ over the Poisson value Γt, but since each cumulant is proportional to t, $P(m;t)$ tends to a Gaussian.[21] To estimate the time scale t^* for the buildup of enhanced cumulants, note that significant enhancement will occur when the oscillator contribution to the variance in Eq. (13a) is larger than the Poisson contribution. From Eq. (14) this requires $(\Gamma\lambda^2 Tt/\hbar\Omega)^2 > \Gamma t$, which yields Eq. (12) for the time scale t^*.

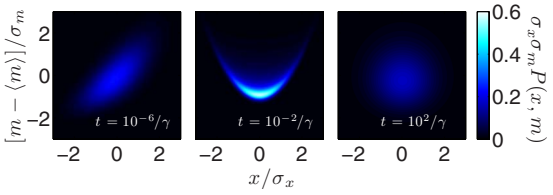

FIG. 2. (Color online) Joint distribution $P(x,m;t)$ calculated for the same times and parameters as in Fig. 1. We have shifted m by its average $\langle m(t) \rangle$ and scaled x and m by their standard deviations, $\sigma_x(t)$ and $\sigma_m(t)$.

In the range of times where the FCS is strongly influenced by the oscillator, $P(m;t)$ is directly related to $P(x)$. For a thermal oscillator at temperature T, we have $P(x) = \sqrt{\frac{\hbar\Omega}{4\pi T}} e^{-\hbar\Omega x^2/4T}$ with x in units of Δx_0. Assuming that fluctuations of x^2 are the dominant source of large m fluctuations, and using Eq. (11), we obtain

$$P(m;t) \propto \exp\left[-\frac{\hbar\Omega m}{4\Gamma\lambda^2 Tt}\right] \quad (15)$$

for $m \gg \Gamma t$. This estimate describes the tail of $P(m;t)$ very well for times $t^* \ll t \ll 1/\gamma$.

IV. CONDITIONAL EVOLUTION

The effects of the oscillator on the FCS are the result of correlations between x^2 and m; we can thus gain further insight by studying conditional dynamics. The joint distribution $P(x,m;t) = \langle x|\hat{\rho}(m;t)|x\rangle$ is shown in Fig. 2. Consistent with the FCS, for short and long times we see only small correlations between x and m. $P(x,m;t)$ is most striking at times $t^* \lesssim t \lesssim 1/\gamma$ due to correlations between x^2 and m.

Equations (8a), (8b), (9a)–(9c), and (10) may also be used to find the conditional energy distribution, $P(E|m;t)$—given a particular measurement history and value of $m(t)$, what is the oscillator's energy distribution? In Fig. 3 we see that for $t^* \ll t \ll 1/\gamma$, the conditional energy distributions are highly nonthermal and localized at the energy required to produce the given value of m from Eq. (11), with width given roughly by T. The ability to obtain information about the oscillator's energy distribution using a weakly coupled detector is some-

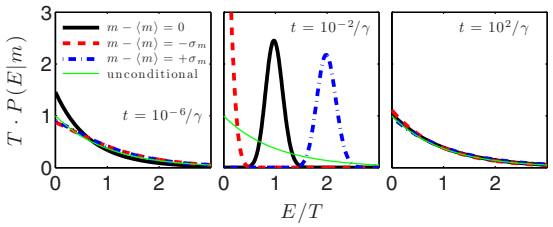

FIG. 3. (Color online) Conditional distributions $P(E|m;t)$ for the same times and parameters as in Fig. 1. The thin line shows the unconditional distribution (average over m).

what surprising and is another result of long-lived energy fluctuations in the oscillator.

V. NON-GAUSSIAN CORRECTIONS TO THE EFFECTIVE BATH MODEL

The effects discussed so far are captured by the effectively classical Liouvillian operator $\mathcal{L}_{\rm cl}$ of Eq. (5). Neglecting the quantum corrections results in an "effective bath" model, where the back-action effects of the tunnel junction are treated as arising from a second thermal bath coupled to the oscillator, and the oscillator is treated as a classical variable which sets the instantaneous tunneling rate. However, the conditional back-action damping and diffusion terms in Eq. (6b) lead to non-Gaussian correlations between the junction current and back-action force operators $\hat{I}$ and $\hat{F}$ that are not captured by the effective bath model. These arise because even though tunneling is stochastic and imparts random momentum kicks to the oscillator, each momentum kick occurs *at the same time* that an electron tunnels. This is completely missed in the effective bath model, as it treats the junction as a thermal noise source independent of individual tunneling events. For example, using Eq. (9) to calculate $\langle\langle x^2(t)m(t)\rangle\rangle$ in the long-time limit, we find an enhancement compared to the effective bath model,

$$\Delta\langle\langle x^2(t)m(t)\rangle\rangle_Q = \frac{\gamma_1 T_1}{\gamma\hbar\Omega} \quad (t\to\infty). \tag{16}$$

This implies the existence of non-Gaussian correlations between the current and back-action force. A direct quantum calculation of the non-Gaussian correlator $\langle\langle\hat{F}(t_1)\hat{F}(t_2)\hat{I}(t_3)\rangle\rangle$ using Keldysh path integrals following Ref. 12 leads to the same non-Gaussian correction given in Eq. (16).

The non-Gaussian correlations may be understood in terms of a simple model of quantum back-action. We describe the oscillator-independent tunneling current as a sequence of delta functions, $I(t)=e\Sigma_{n=0}^{\infty}\delta(t-t_n)$, where the intervals between the t_n are exponentially distributed. The back-action force of the junction is then taken to be $F(t)=\Sigma_{n=0}^{\infty}\xi_n\delta(t-t_n)$, where ξ_n is a zero-mean random variable describing the impulse imparted to the oscillator by the nth electron. The same sequence of times $\{t_n\}$ appears in both $I(t)$ and $F(t)$, reflecting the fact that back-action arises from the action of individual tunneling electrons. If we then take $\langle\xi_m\xi_n\rangle=(\hbar\tau_1/\tau_0)^2\delta_{mn}$, our simple model reproduces the non-Gaussian correlations obtained from Eqs. (8a), (8b), (9a)–(9c), and (10); we also obtain the expected back-action diffusion constant D_1. From the size of ξ_n we see that the typical momentum kick imparted by a single tunneling electron is given by $\Delta p\sim\hbar\tau_1/\tau_0$ and not by the Fermi momentum. This value for Δp is consistent with the Heisenberg uncertainty principle since the sensitivity of a position measurement scales as $\Delta x\sim\tau_0/\tau_1$. We thus have a simple picture for the source of the conditional part of quantum back-action: it arises from tunneling electrons imparting random momentum kicks of size set by the uncertainty principle. Again, we stress that this picture only accounts for the conditional half of the back-action damping and diffusion; the other half is completely uncorrelated with tunneling electrons [cf. Eq. (6b) and the discussion thereafter]. We also note that one can derive the conditional back-action terms in Eq. (6b) directly from this simple model, from a corresponding classical master equation in which each tunneling event is associated with a random momentum kick.

The non-Gaussian correlations discussed above can in principle be detected via the finite-frequency current noise in the tunnel junction, $S_I(\omega)$. This may be found from the time dependence of $\langle\langle m^2\rangle\rangle$ using the MacDonald formula,[22]

$$S_I(\omega) = 2e^2\omega\int_0^{\infty} dt\,\sin(\omega t)\partial_t\langle\langle m^2(t)\rangle\rangle. \tag{17}$$

Note that the frequency-dependent current noise is obtained from the *particle* current fluctuations only. In the single junction, tunneling is nonresonant and there is no place for charge to build up in the system, so displacement currents may be safely neglected.[23] The time derivative of $\langle\langle m^2\rangle\rangle$ is given in Eq. (13a), which shows that we need the full time-dependent correlations $\langle\langle x(t)m(t)\rangle\rangle$ and $\langle\langle x^2(t)m(t)\rangle\rangle$ to calculate the current noise. These correlations are calculated simply by taking the χ derivative of Eqs. (8a), (8b), (9a)–(9c), and (10). The resulting equations are readily solved exactly for the physical initial conditions in which the oscillator is equilibrated with both the environment and the tunnel junction, i.e., $\langle\langle x^2\rangle\rangle=\langle\langle p^2\rangle\rangle=2T$ in our units. The correlation $\langle\langle xm\rangle\rangle$ leads to a peak at $\omega=\Omega$ in the noise that is very accurately captured by the effective bath model. However, the correlation $\langle\langle x^2m\rangle\rangle$ leads to peaks at $\omega=0$ and $\omega=2\Omega$ that show signatures of the non-Gaussian correlations. This is especially true in the limit $\gamma_0/\gamma_1\gg T_1/T_0\gg 1$, where the non-Gaussian correlations lead to a doubling of the current noise peak at $\omega=0$ and completely suppress the peak at $\omega=2\Omega$ as shown in Fig. 4. This limit requires a back-action damping rate much smaller than the intrinsic damping from the environment and a back-action temperature much greater than the temperature of the environment. The first of these conditions is natural in experiments and the second has been achieved. Note that we still require the intrinsic damping of the oscillator to be small.

Measurements of $S_I(\omega)$ could thus be used to distinguish the tunnel junction's back-action on the oscillator from the effects of a Gaussian uncorrelated noise source supplied by an equilibrium bath. However, as seen in Fig. 4, in the same

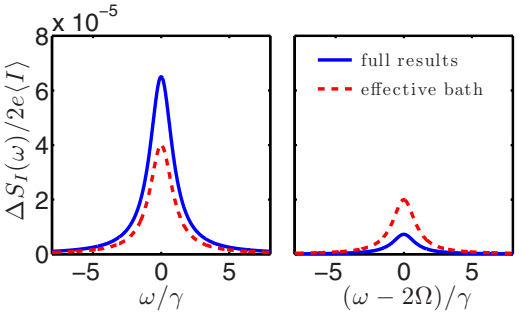

FIG. 4. (Color online) Contribution to the current noise spectrum near $\omega=0$ (left) and $\omega=2\Omega$ (right) from to the correlation $\langle\langle x^2 m\rangle\rangle$. The full calculation including non-Gaussian corrections (solid) is compared to the results from the effective bath model (dashed). The contributions are normalized by the frequency-independent shot-noise background. We have taken $T_0=\hbar\Omega/2$, $T_1=100\hbar\Omega$, $\lambda=0.01$, $\tau_0=0.2$, and $\gamma_0=10^{-3}\Omega$ in order to approach the limit where the non-Gaussian signatures are maximized.

limit where the signatures are relatively large, the peak heights themselves are very small compared to the frequency-independent shot-noise background. For this reason, detecting these signatures in the current noise would pose a formidable challenge. Further thought will be devoted to more efficient strategies to detect the non-Gaussian correlations we have identified.

VI. CONCLUSIONS

We have studied the statistics of the experimentally relevant NEMS of a tunnel junction coupled to a mechanical oscillator. We have shown that even if the coupling is very weak, long-lived energy fluctuations in the oscillator allow it to dominate the FCS. The oscillator-induced enhancement of the third moment of the FCS could be observed up to measurement times near $1/\gamma$, well within reach of current experiments. We have also shown that the effective bath model is not sufficient to fully describe the effects of the tunnel junction on the oscillator. Half of the back-action is conditional as a result of the random momentum kicks imparted to the oscillator by tunneling electrons, and this leads to non-Gaussian correlations with signatures in the finite-frequency current noise.

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[20] Including the quadratic x dependence in the amplitude leads to a renormalization of λ as well as two-phonon absorption or emission during a tunneling event; both are unimportant in the limit of weak coupling.
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